Periodic Solutions to Painlevé VI and Dynamical System on Cubic Surface

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Dedicated to Professor Masuo Hukuhara on his 100th birthday

Abstract

The number of periodic solutions to Painlevé VI along a Pochhammer loop is counted exactly. It is shown that the number grows exponentially with period, where the growth rate is determined explicitly. Principal ingredients of the computation are a moduli-theoretical formulation of Painlevé VI, a Riemann-Hilbert correspondence, the dynamical system of a birational map on a cubic surface, and the Lefschetz fixed point formula.

1 Introduction

Painlevé equations and dynamical systems on complex surfaces are two subjects of mathematics which have been investigated actively in recent years. In this paper we shall demonstrate a substantial relation between them by presenting a fruitful application to the former subject of the latter. We begin with stating our motivation on the side of Painlevé equations.

The global structure of the sixth Painlevé equation $\mathrm{P_{VI}}(\kappa)$, especially the multivalued character of its solutions is an important issue in the study of Painlevé equations. In this respect, several authors [3, 4, 6, 11, 12, 20, 21] have been interested in finding algebraic solutions, because they offer a simplest class of solutions with clear global structure in the sense that they have only finitely many branches under analytic continuations along all loops in the domain

$$X = \mathbb{P}^1 - \{0, 1, \infty\}. \tag{1}$$

In another direction of promising research, we are interested in periodic solutions along a single loop, namely, in those solutions which are finitely many-valued along a single loop chosen particularly. Given such a loop, we shall discuss the following problems:

- How many solutions can be periodic of period N among all solutions to $\mathrm{P_{VI}}(\kappa)$?
- How rapidly does that number grow as the period N tends to infinity?

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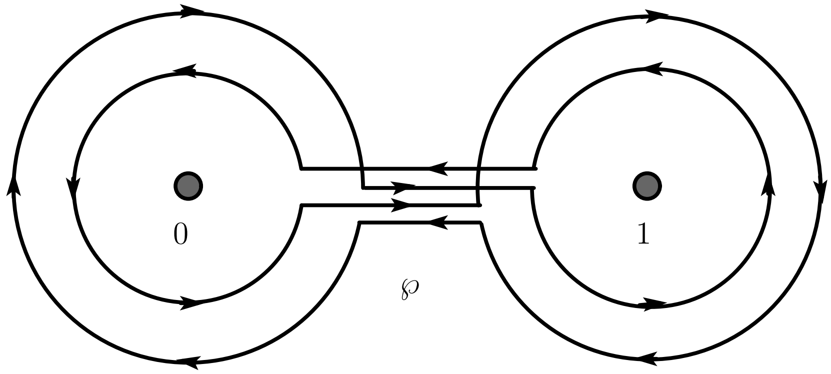

Figure 1: Pochhammer loop $\wp$

For such a loop we take a Pochhammer loop $\wp$ as in Figure 1. If ℓ_0 and ℓ_1 are standard generators of $\pi_1(X,x)$ as in Figure 2, then $\wp$ is a loop homotopic to the commutator

$$[\ell_0, \ell_1^{-1}] = \ell_0 \ell_1^{-1} \ell_0^{-1} \ell_1.$$

It is a typical loop which often appears in mathematics due to the property that any abelian representation of $\pi_1(X,x)$ is killed along this loop. For example, it is used as an integration contour of Euler integral representation of hypergeometric functions [18]. In the context of this article the Pochhammer loop will be closely related to a certain birational map of a cubic surface whose dynamics is quite relevant to understanding the global structure of the sixth Painlevé equation (see discussions in Sections 6 and 7).

We remark that the same problem for a simplest loop, namely for a loop ℓ_0 or ℓ_1 in Figure 2 or a loop $\ell_\infty = (\ell_0 \ell_1)^{-1}$ around the point at infinity, is not interesting or even meaningless, because for any $N > 1$, there are infinitely many periodic solutions of period N along it. In fact it turns out that they are parametrized by points on certain complex curves and hence their cardinality is that of a continuum [17]. On the contrary, along the Pochhammer loop $\wp$, the cardinality of the periodic solutions of period N turns out to be finite for every $N \in \mathbb{N} := \{1, 2, 3, \dots\}$ and hence our problem certainly makes sense.

In this article we shall exactly count the number of periodic solutions to $\mathrm{P_{VI}}(\kappa)$ along the Pochhammer loop $\wp$ under a certain generic assumption on the parameters κ. In particular we shall show that the number grows exponentially as the period tends to infinity, with the growth rate determined explicitly (see Theorem 2.1). As is already mentioned, this result is the fruits

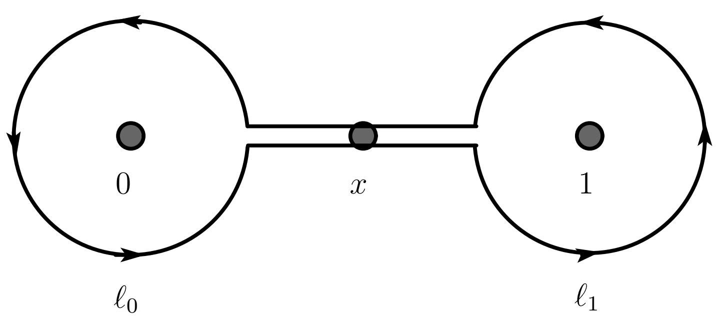

Figure 2: Standard generators of $\pi_1(X,x)$

of a good application to Painlevé equations of a dynamical system theory on complex surfaces as developed in [5, 7]. Algebraic geometry of Painlevé equations, especially a moduli-theoretical interpretation of Painlevé dynamics [13, 14] is also an essential ingredient of our work.

After stating the main result of this article in Section 2, we shall develop the story of this article in the following manner. First, $\mathrm{P_{VI}}(\kappa)$ is formulated as a flow, *Painlevé flow*, on a moduli space of stable parabolic connections (Section 3). Secondly, it is conjugated to an isomonodromic flow on a moduli space of monodromy representations via a Riemann-Hilbert correspondence (Section 4). Thirdly, with a natural identification of the representation space with a cubic surface, the Poincaré section of $\mathrm{P_{VI}}(\kappa)$ is conjugated to the dynamical system of a group action on the cubics (Section 5). Especially, analytic continuation along the Pochhammer loop is connected with a distinguished transformation, called a *'Coxeter' transformation*, of the group action. Fourthly, main properties of our dynamical system on the cubics are established from the standpoint of birational surface dynamics. Fifthly, the number of periodic points of the Coxeter transformation is counted by using the Lefschetz fixed point formula (Section 8). Then, back to the original phase space of $\mathrm{P_{VI}}(\kappa)$, we arrive at our final goal, that is, the exact number of periodic solutions to $\mathrm{P_{VI}}(\kappa)$ of any period along the Pochhammer loop $\wp$.

The authors would be happy if this article could give a new insight into the global structure of the sixth Painlevé equation. They are grateful to Yutaka Ishii for valuable discussions.

2 Main Result

Let us describe our main result in more detail. To this end we recall that the sixth Painlevé equation $\mathrm{P_{VI}}(\kappa)$ in its Hamiltonian form is a system of nonlinear differential equations

$$\frac{dq}{dx} = \frac{\partial H(\kappa)}{\partial p}, \qquad \frac{dp}{dx} = -\frac{\partial H(\kappa)}{\partial q}, \tag{2}$$

with an independent variable $x \in X$ and unknown functions $(q(x), p(x))$, depending on complex parameters $\kappa = (\kappa_0, \kappa_1, \kappa_2, \kappa_3, \kappa_4)$ in a four-dimensional affine space

$$\mathcal{K} := \{\, \kappa = (\kappa_0, \kappa_1, \kappa_2, \kappa_3, \kappa_4) \in \mathbb{C}^5 \,:\, 2\kappa_0 + \kappa_1 + \kappa_2 + \kappa_3 + \kappa_4 = 1 \,\},$$

where the Hamiltonian $H(\kappa) = H(q, p, x; \kappa)$ is given by

$$x(x-1)H(\kappa) \;=\; (q_0 q_1 q_x)p^2 - \{\kappa_1 q_1 q_x + (\kappa_2 - 1) q_0 q_1 + \kappa_3 q_0 q_x\} p + \kappa_0(\kappa_0 + \kappa_4) q_x,$$

with $q_\nu = q - \nu$ for $\nu \in \{0, 1, x\}$. It is known that system (2) enjoys the Painlevé property, that is, any meromorphic solution germ at a base point $x \in X$ of system (2) admits a global analytic continuation along any path emanating from x as a meromorphic function.

Geometrically, the sixth Painlevé equation $\mathrm{P_{VI}}(\kappa)$ is formulated as a holomorphic uniform foliation on the total space of a fibration of certain smooth, quasi-projective, rational surfaces,

$$\pi_\kappa : M(\kappa) \to X, \tag{3}$$

transversal to each fiber of the fibration. We refer to [1, 13, 14, 23, 24, 25] for the detailed accounts of the space $M(\kappa)$. Especially the papers [13, 14] give a comprehensive description of it as a moduli space of stable parabolic connections. The fiber $M_x(\kappa)$ over $x \in X$, called

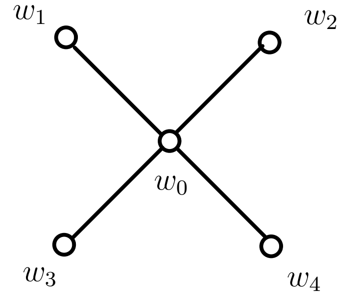

Figure 3: Dynkin diagram of type $D_4^{(1)}$

the space of initial conditions at time x, parametrizes all the solution germs at x of equation (2) most precisely, completing the naïve and incomplete space $\mathbb{C}^2$ of initial values (q,p) at the points x. Given a loop $\gamma \in \pi_1(X,x)$, the horizontal lifts of the loop γ along the foliation induces a biholomorphism $\gamma_* : M_x(\kappa) \to M_x(\kappa)$, called the Poincaré return map along γ, which depends only on the homotopy class of γ. Then the global structure of the sixth Painlevé equation $\mathrm{P_{VI}}(\kappa)$ is described by the group homomorphism

$$\mathrm{PS}_x(\kappa) \ : \ \pi_1(X,x) \to \mathrm{Aut}\, M_x(\kappa), \qquad \gamma \mapsto \gamma_*, \tag{4}$$

which is referred to as the *Poincaré section* of the sixth Painlevé dynamics $\mathrm{P_{VI}}(\kappa)$.

In this article we are interested in analytic continuations of solutions to equation (2) along the Pochhammer loop $\wp$, namely, in the iteration of the Poincaré return map $\wp_*$ along $\wp$. The Poincaré return map along the Pochhammer loop is referred to as the *Pochhammer-Poincaré map*. Given any $N \in \mathbb{N}$, let $\mathrm{Per}_N(\kappa)$ be the set of all initial points $Q \in M_x(\kappa)$ that come back to the original positions after the N-th iterate of the Pochhammer-Poincaré map $\wp_*$,

$$\mathrm{Per}_N(\kappa) := \{\, Q \in M_x(\kappa) \ : \ \wp_*^N(Q) = Q \,\}. \tag{5}$$

The aim of this article is to count the number of $\mathrm{Per}_N(\kappa)$ and to find out its growth rate as the period N tends to infinity.

To avoid certain technical difficulties (see Remark 2.4), we make a generic assumption on the parameters $\kappa \in \mathcal{K}$. To this end we recall an affine Weyl group structure of the parameter space [13, 16]. The affine space $\mathcal{K}$ is identified with the linear space $\mathbb{C}^4$ by the isomorphism

$$\mathcal{K} \to \mathbb{C}^4, \quad \kappa = (\kappa_0,\kappa_1,\kappa_2,\kappa_3,\kappa_4) \mapsto (\kappa_1,\kappa_2,\kappa_3,\kappa_4),$$

where the latter space $\mathbb{C}^4$ is equiped with the standard (complex) Euclidean inner product. For each $i \in \{0,1,2,3,4\}$, let $w_i : \mathcal{K} \to \mathcal{K}$ be the orthogonal reflection having $\{\kappa_i = 0\}$ as its reflecting hyperplane, with respect to the inner product mentioned above. Then the group generated by w_0, w_1, w_2, w_3, w_4 is an affine Weyl group of type $D_4^{(1)}$,

$$W(D_4^{(1)}) = \langle w_0, w_1, w_2, w_3, w_4 \rangle \curvearrowright \mathcal{K}.$$

corresponding to the Dynkin diagram in Figure 3. The reflecting hyperplanes of all reflections in the group $W(D_4^{(1)})$ are given by affine linear relations

$$\kappa_i = m, \qquad \kappa_1 \pm \kappa_2 \pm \kappa_3 \pm \kappa_4 = 2m+1 \qquad (i \in \{1,2,3,4\},\ m \in \mathbb{Z}),$$

with any choice of signs $\pm$. Let **Wall** be the union of all those hyperplanes. Then the generic condition to be imposed on parameters is that κ should lie outside **Wall**; this is a necessary and sufficient condition for $P_{VI}(\kappa)$ to admit no Riccati solutions [13].

Now the main theorem of this article is stated as follows.

Theorem 2.1 *For any* $\kappa \in \mathcal{K} - \mathbf{Wall}$ *the cardinality of the set* $\mathrm{Per}_N(\kappa)$ *is given by*

$$\#\,\mathrm{Per}_N(\kappa) = (9+4\sqrt{5})^N + (9-4\sqrt{5})^N + 4 \qquad (N \in \mathbb{N}). \tag{6}$$

Remark 2.2 It should be noted that formula (6) is rewritten as

$$\#\,\mathrm{Per}_N(\kappa) - \{(9+4\sqrt{5})^N + 4\} = (9+4\sqrt{5})^{-N},$$

which means that the geometric sequence $(9+4\sqrt{5})^N$ shifted by 4 approximates the cardinality of $\mathrm{Per}_N(\kappa)$ up to an exponentially decaying error term $(9+4\sqrt{5})^{-N}$, where the growth rate of cardinality and the decay rate of error term are given by the same number $9+4\sqrt{5}$. Moreover, since $9 \pm 4\sqrt{5}$ are the root of the quadratic equation $\lambda^2 - 18\lambda + 1 = 0$, the formula (6) is expressed as $\#\,\mathrm{Per}_N(\kappa) = C_N + 4$, where the sequence $\{C_N\}$ is defined recursively by

$$C_0 = 2, \qquad C_1 = 18, \qquad C_{N+2} - 18\,C_{N+1} + C_N = 0.$$

Remark 2.3 Our main theorem can also be stated in terms of a dynamical zeta function. Indeed, as a generating expression of formula (6) for all $N \in \mathbb{N}$, we have

$$Z_\kappa(z) := \exp\left(\sum_{N=1}^{\infty} \#\,\mathrm{Per}_N(\kappa)\,\frac{z^N}{N}\right) = \frac{1}{(1-z)^4(1-18z+z^2)}.$$

Remark 2.4 In this article we restrict our attention to the generic case $\kappa \in \mathcal{K} - \mathbf{Wall}$ only, leaving the nongeneric case $\kappa \in \mathbf{Wall}$ untouched. The difference between the generic case and the nongeneric case lies in the fact that the Riemann-Hilbert correspondence to be used in the proof becomes a biholomorphism in the former case, while it gives an analytic minimal resolution of Klein singularities in the latter case (see Remark 4.2). The presence of singularities would make the treatment of the nongeneric case more complicated. However it is expected that the basic strategy developed in this article will also be effective in the nongeneric case. The relevant discussion will be made in another place.

3 Moduli Space of Stable Parabolic Connections

In order to describe the fibration (3), we first construct an auxiliary fibration $\pi_\kappa : \mathcal{M}(\kappa) \to T$ over the configuration space of mutually distinct, ordered, three points in $\mathbb{C}$,

$$T = \{\, t = (t_1, t_2, t_3) \in \mathbb{C}^3 \,:\, t_i \neq t_j \text{ for } i \neq j \,\},$$

and then reduce it to the original fibration (3). We put the fourth point t_4 at infinity. Given any $(t, \kappa) \in T \times \mathcal{K}$, a (t, κ)-parabolic connection is a quadruple $Q = (E, \nabla, \psi, l)$ such that

(1) E is a rank 2 vector bundle of degree -1 over $\mathbb{P}^1$,

singularities	t_1	t_2	t_3	t_4
first exponent	$-\lambda_1$	$-\lambda_2$	$-\lambda_3$	$-\lambda_4$
second exponent	λ_1	λ_2	λ_3	$\lambda_4 - 1$
difference	κ_1	κ_2	κ_3	κ_4

Table 1: Riemann scheme: κ_i is the difference of the second exponent from the first.

(2) $\nabla : E \to E \otimes \Omega^1_{\mathbb{P}^1}(D_t)$ is a Fuchsian connection with pole divisor $D_t = t_1 + t_2 + t_3 + t_4$ and Riemann scheme as in Table 1, where $t_4 = \infty$ as mentioned above,

(3) $\psi : \det E \to \mathcal{O}_{\mathbb{P}^1}(-t_4)$ is a horizontal isomorphism called a determinantal structure, where $\mathcal{O}_{\mathbb{P}^1}(-t_4)$ is equiped with the connection induced from $d : \mathcal{O}_{\mathbb{P}^1} \to \Omega^1_{\mathbb{P}^1}$,

(4) $l = (l_1, l_2, l_3, l_4)$ is a parabolic structure, namely, l_i is an eigenline of $\mathrm{Res}_{t_i}(\nabla) \in \mathrm{End}(E_{t_i})$ corresponding to eigenvalue λ_i (whose minus is the first exponent $-\lambda_i$ in Table 1).

There exists a concept of stability for parabolic connections, with which the geometric invariant theory [22] can be worked out to establish the following theorem [13, 14].

Theorem 3.1 *For any $(t, \kappa) \in T \times \mathcal{K}$ there exists a fine moduli scheme $\mathcal{M}_t(\kappa)$ of stable (t, κ)-parabolic connections. The moduli space $\mathcal{M}_t(\kappa)$ is a smooth, irreducible, quasi-projective surface. As a relative setting over T, for any $\kappa \in \mathcal{K}$, there exists a family of moduli spaces*

$$\pi_\kappa : \mathcal{M}(\kappa) \to T \tag{7}$$

such that the projection π_κ is a smooth morphism with fiber $\mathcal{M}_t(\kappa)$ over $t \in T$.

Now the fibration (3) is defined to be the pull-back of (7) by an injection

$$\iota : X \hookrightarrow T, \quad x \mapsto (0, x, 1),$$

The group $\mathrm{Aff}(\mathbb{C})$ of affine linear transformations on $\mathbb{C}$ acts diagonally on the configuration space T and the quotient space $T/\mathrm{Aff}(\mathbb{C})$ is isomorphic to X, with the quotient map given by

$$r : T \to X, \quad t = (t_1, t_2, t_3) \mapsto x = \frac{t_2 - t_1}{t_3 - t_1}. \tag{8}$$

The map r yields a trivial $\mathrm{Aff}(\mathbb{C})$-bundle structure of T over X and the fibration (7) is in turn the pull-back of the fibration (3) by the map r. Hence we have a commutative diagram

$$\begin{tikzcd} \mathcal{M}(\kappa) \arrow[r] \arrow[d, "\pi_\kappa"'] & M(\kappa) \arrow[d, "\pi_\kappa"] \\ T \arrow[r, "r"'] & X. \end{tikzcd} \tag{9}$$

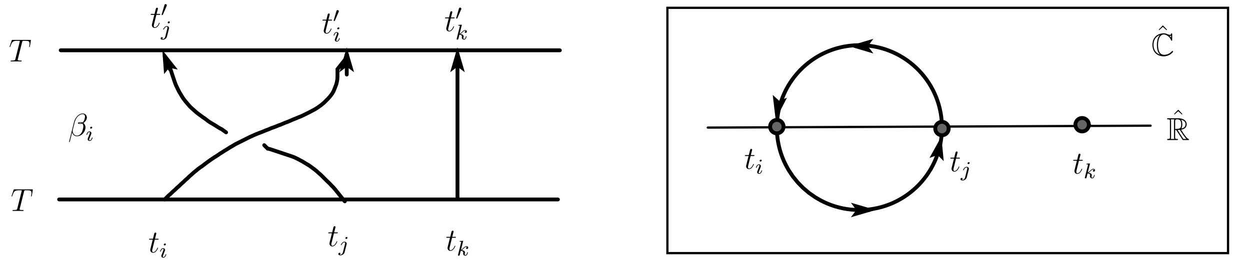

Figure 4: Basic braid β_i in T and the corresponding movement of t in $\hat{\mathbb{C}}$

In [13, 14] the Painlevé dynamics $\mathrm{P_{VI}}(\kappa)$ is formulated as a holomorphic uniform foliations on the fibration (7) which is compatible with the diagram (9). Thus the Poincaré section (4) is reformulated as a group homomorphism

$$\mathrm{PS}_t(\kappa) \ : \ \pi_1(T,t) \to \mathrm{Aut}\,\mathcal{M}_t(\kappa). \tag{10}$$

Let us describe the fundamental group $\pi_1(T,t)$ in terms of a braid group [2]. We take a base point $t = (t_1, t_2, t_3) \in T$ in such a manner that the three points lie on the real line in an increasing order $t_1 < t_2 < t_3$. To treat them symmetrically, we denote them by t_i, t_j, t_k for a cyclic permutation (i, j, k) of $(1, 2, 3)$ and think of them as cyclically ordered three points on the equator $\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ of the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Let β_i be a braid on three strings as in Figure 4 (left) along which t_i and t_j make a half-turn, with t_i moving in the southern hemisphere and t_j in the northern hemisphere, while t_k is kept fixed as in Figure 4 (right). Then the braid group on three strings is the group generated by β_i, β_j and β_k and the pure braid group P_3 is the normal subgroup of B_3 generated by the squares β_i^2, β_j^2 and β_k^2,

$$P_3 = \langle \beta_i^2, \beta_j^2, \beta_k^2 \rangle \lhd B_3 = \langle \beta_i, \beta_j, \beta_k \rangle.$$

The generators of B_3 satisfy relations $\beta_i\beta_j\beta_i = \beta_j\beta_i\beta_j$ and $\beta_k = \beta_i\beta_j\beta_i^{-1}$, and so B_3 is generated by β_i and β_j only. The fundamental group $\pi_1(T,t)$ is identified with the pure braid group P_3.

The reduction map (8) induces a group homomorphism $r_* : \pi_1(T,t) \to \pi_1(X,x)$. It is easy to see that the loops ℓ_0 and ℓ_1 in Figure 2 are the r_*-images of β_1^2 and β_2^2 respectively, so that the Pochhammer loop $\wp$ in X is the r_*-image of the pure braid

$$[\beta_1^2, \beta_2^{-2}] = \beta_1^2\beta_2^{-2}\beta_1^{-2}\beta_2^2. \tag{11}$$

Thus we will be concerned with the Poincaré section (10) along this particular braid.

The symmetric group S_3 acts on T by permuting the entries of $t = (t_1, t_2, t_3)$ and the quotient space T/S_3 is the configuration space of mutually distinct, unordered, three points in $\mathbb{C}$. The fundamental group $\pi(T/S_3, s)$ with base point $s = \{t_1, t_2, t_3\}$ is identified with the ordinary braid group B_3 and there exists a short exact sequence of groups

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1(T,t) & \longrightarrow & \pi_1(T/S_3,s) & \longrightarrow & S_3 & \longrightarrow & 1 \\
 & & \Vert & & \Vert & & \Vert & & \\
1 & \longrightarrow & P_3 & \longrightarrow & B_3 & \longrightarrow & S_3 & \longrightarrow & 1.
\end{array}$$

Then the Poincaré section (10) naturally lifts to a collection of isomorphisms

$$\beta_* : \mathcal{M}_t(\kappa) \to \mathcal{M}_{\tau(t)}(\tau(\kappa)), \qquad (\beta \in B_3)$$

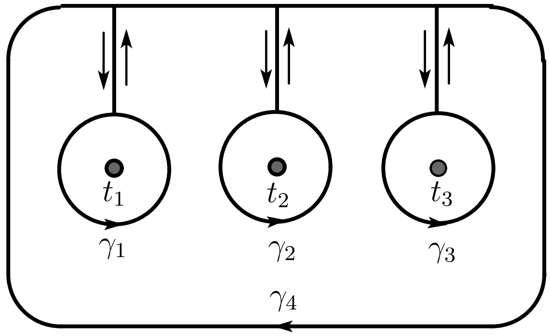

Figure 5: Four loops in $\mathbb{P}^1 - D_t$; the fourth point t_4 is outside γ_4, invisible.

which may be called the *half-Poincaré section* of $\mathrm{P_{VI}}(\kappa)$, where $\tau \in S_3$ denotes the permutation corresponding to $\beta \in B_3$. Note that $\tau \in S_3$ acts on $\kappa \in \mathcal{K}$ too by permuting the entries of $(\kappa_1, \kappa_2, \kappa_3)$ in the same manner as it does on $t = (t_1, t_2, t_3)$, since κ_i is loaded on t_i. Now the permutation corresponding to the basic braid β_i is the substitution $\tau_i = (i, j)$ that exchanges t_i and t_j while keeping t_k fixed. Thus there are three basic half-Poincaré maps:

$$\beta_{i*} : \mathcal{M}_t(\kappa) \to \mathcal{M}_{\tau_i(t)}(\tau_i(\kappa)), \qquad (i = 1, 2, 3). \tag{12}$$

4 Riemann-Hilbert Correspondence

It is rather hopeless to deal with the Painlevé flow directly, since it is a highly transcendental dynamical system on the moduli space of stable parabolic connections. But it can be recast into a more tractable dynamical system, called an isomonodromic flow, on a moduli space of monodromy representations via a Riemann-Hilbert correspondence. We review the construction of such a Riemann-Hilbert correspondence in the sequel.

Let $A := \mathbb{C}^4$ be the complex 4-space with coordinates $a = (a_1, a_2, a_3, a_4)$, called the space of local monodromy data. Given $(t, a) \in T \times A$, let $\mathcal{R}_t(a)$ be the moduli space of Jordan equivalence classes of representations $\rho : \pi_1(\mathbb{P}^1 - D_t, *) \to SL_2(\mathbb{C})$ such that $\operatorname{Tr} \rho(\gamma_i) = a_i$ for $i \in \{1, 2, 3, 4\}$, where the divisor $D_t = t_1 + t_2 + t_3 + t_4$ is identified with the point set $\{t_1, t_2, t_3, t_4\}$ and γ_i is a loop surrounding t_i as in Figure 5. Any stable parabolic connection $Q = (E, \nabla, \psi, l) \in \mathcal{M}_t(\kappa)$, when restricted to $\mathbb{P}^1 - D_t$, induces a flat connection

$$\nabla|_{\mathbb{P}^1 - D_t} : E|_{\mathbb{P}^1 - D_t} \to (E|_{\mathbb{P}^1 - D_t}) \otimes \Omega^1_{\mathbb{P}^1 - D_t},$$

and one can speak of the Jordan equivalence class ρ of its monodromy representations. Then the Riemann-Hilbert correspondence at $t \in T$ is defined by

$$\mathrm{RH}_{t,\kappa} : \mathcal{M}_t(\kappa) \to \mathcal{R}_t(a), \quad Q \mapsto \rho, \tag{13}$$

where in view of the Riemann scheme in Table 1, the local monodromy data $a \in A$ is given by

$$a_i = \begin{cases} 2\cos \pi\kappa_i & (i = 1, 2, 3), \\ -2\cos \pi\kappa_4 & (i = 4). \end{cases} \tag{14}$$

As a relative setting over T, let $\pi_a : \mathcal{R}(a) \to T$ be the family of moduli spaces of monodromy representations with fiber $\mathcal{R}_t(a)$ over $t \in T$. Then the relative version of Riemann-Hilbert correspondence is formulated to be the commutative diagram

$$\begin{array}{ccc} \mathcal{M}(\kappa) & \xrightarrow{\mathrm{RH}_\kappa} & \mathcal{R}(a) \\ {\scriptstyle \pi_\kappa}\big\downarrow & & \big\downarrow{\scriptstyle \pi_a} \\ T & = & T, \end{array} \tag{15}$$

whose fiber over $t \in T$ is given by (13). Then we have the following theorem [13, 14].

Theorem 4.1 *If $\kappa \in \mathcal{K} - \mathbf{Wall}$, then $\mathcal{R}(a)$ as well as each fiber $\mathcal{R}_t(a)$ is smooth and the Riemann-Hilbert correspondence RH_κ in (15) is a biholomorphism.*

Remark 4.2 If $\kappa \in \mathbf{Wall}$, then $\mathcal{R}_t(a)$ is not a smooth surface but a surface with Klein singularities and (13) yields an analytic minimal resolution of singularities, so that (15) gives a family of resolutions of singularities [13]. As is mentioned in Remark 2.4, this fact makes the treatment of the nongeneric case more involved and we leave this case in another occasion.

5 Cubic Surface and the 27 Lines

The moduli space $\mathcal{R}_t(a)$ of monodromy representations is isomorphic to an affine cubic surface $\mathcal{S}(\theta)$ and the braid group action on $\mathcal{R}_t(a)$ can be made explicit in terms of $\mathcal{S}(\theta)$. Let us recall this construction [13]. Given $\theta = (\theta_1, \theta_2, \theta_3, \theta_4) \in \Theta := \mathbb{C}^4_\theta$, consider an affine cubic surface

$$\mathcal{S}(\theta) = \{\, x = (x_1, x_2, x_3) \in \mathbb{C}^3_x \,:\, f(x, \theta) = 0 \,\},$$

where the cubic polynomial $f(x, \theta)$ of x with parameter θ is given by

$$f(x, \theta) = x_1 x_2 x_3 + x_1^2 + x_2^2 + x_3^2 - \theta_1 x_1 - \theta_2 x_2 - \theta_3 x_3 + \theta_4.$$

Then there exists an isomorphism of affine algebraic surfaces, $\mathcal{R}_t(a) \to \mathcal{S}(\theta)$, $\rho \mapsto x$, where

$$x_i = \mathrm{Tr}\, \rho(\gamma_j \gamma_k), \qquad \text{for} \quad \{i, j, k\} = \{1, 2, 3\},$$

together with a correspondence of parameters, $A \to \Theta$, $a \mapsto \theta$, given by

$$\theta_i = \begin{cases} a_i a_4 + a_j a_k & (\{i, j, k\} = \{1, 2, 3\}), \\ a_1 a_2 a_3 a_4 + a_1^2 + a_2^2 + a_3^2 + a_4^2 - 4 & (i = 4). \end{cases} \tag{16}$$

With this identification, the Riemann-Hilbert correspondence (13) is reformulated as a map

$$\mathrm{RH}_t(\kappa) : \mathcal{M}_t(\kappa) \to \mathcal{S}(\theta), \qquad \text{with} \quad \theta = \mathrm{rh}(\kappa), \tag{17}$$

where $\mathrm{rh} : \mathcal{K} \to \Theta$ is the composition of two maps $\mathcal{K} \to A$ and $A \to \Theta$ defined by (14) and (16), which we call the Riemann-Hilbert correspondence in the parameter level. Through the reformulated Riemann-Hilbert correspondence (17), the i-th basic half-Poincaré map β_{i*} in (12) is conjugated to a map $g_i : \mathcal{S}(\theta) \to \mathcal{S}(\theta')$, $(x, \theta) \mapsto (x', \theta')$, which is explicitly represented as

$$g_i \quad : \quad (x_i', x_j', x_k', \theta_i', \theta_j', \theta_k', \theta_4') = (\theta_j - x_j - x_k x_i, x_i, x_k, \theta_j, \theta_i, \theta_k, \theta_4). \tag{18}$$

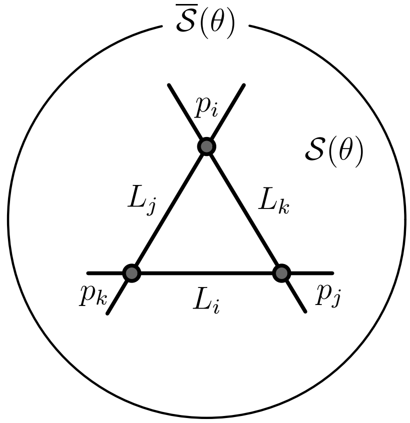

Figure 6: Tritangent lines at infinity on $\overline{\mathcal{S}}(\theta)$

A derivation of this formula can be found in [16] (see also [4, 6, 9, 10, 15, 19]). By Theorem 4.1 the map (17) is an isomorphism and hence (18) is a strict conjugacy of (12). We can easily check the relations $g_i g_j g_i = g_j g_i g_j$ and $g_k = g_i g_j g_i^{-1}$ which are parallel to those for β_i, β_j, β_k.

To utilize standard techniques from algebraic geometry and complex geometry, we need to compactify the affine cubic surface $\mathcal{S}(\theta)$ by a standard embedding

$$\mathcal{S}(\theta) \hookrightarrow \overline{\mathcal{S}}(\theta) \subset \mathbb{P}^3, \qquad x = (x_1, x_2, x_3) \mapsto [1 : x_1 : x_2 : x_3],$$

where the compactified surface $\overline{\mathcal{S}}(\theta)$ is defined by $\overline{\mathcal{S}}(\theta) = \{ X \in \mathbb{P}^3 \,:\, F(X, \theta) = 0 \}$ with

$$F(X, \theta) = X_1 X_2 X_3 + X_0(X_1^2 + X_2^2 + X_3^2) - X_0^2(\theta_1 X_1 + \theta_2 X_2 + \theta_3 X_3) + \theta_4 X_0^3.$$

It is obtained from the affine surface $\mathcal{S}(\theta)$ by adding three lines at infinity,

$$L_i = \{ X \in \mathbb{P}^3 \,:\, X_0 = X_i = 0 \} \qquad (i = 1, 2, 3). \tag{19}$$

Here and hereafter the homogeneous coordinates $X = [X_0 : X_1 : X_2 : X_3]$ of $\mathbb{P}^3$ should not be confused with the domain X in (1). The union $L = L_1 \cup L_2 \cup L_3$ is called the tritangent lines at infinity and the intersection point of L_j and L_k is denoted by p_i (see Figure 6). Note that

$$p_1 = [0 : 1 : 0 : 0], \qquad p_2 = [0 : 0 : 1 : 0], \qquad p_3 = [0 : 0 : 0 : 1].$$

For $i \in \{1, 2, 3\}$, put $U_i = \{ X \in \mathbb{P}^3 \,:\, X_i \neq 0 \}$ and take inhomogeneous coordinates of $\mathbb{P}^3$;

$$\begin{array}{rclcll} u & = & (u_0, u_j, u_k) & = & (X_0/X_i, X_j/X_i, X_k/X_i) & \text{on} \quad U_i, \\ v & = & (v_0, v_i, v_k) & = & (X_0/X_j, X_i/X_j, X_k/X_j) & \text{on} \quad U_j, \\ w & = & (w_0, w_i, w_j) & = & (X_0/X_k, X_i/X_k, X_j/X_k) & \text{on} \quad U_k, \end{array} \tag{20}$$

where $\{i, j, k\} = \{1, 2, 3\}$. In terms of these coordinates we shall find local coordinates and local equations of $\overline{\mathcal{S}}(\theta)$ around L. Since $L \subset U_1 \cup U_2 \cup U_3$, we can divide L into components $L \cap U_i$, $i = 1, 2, 3$, and make a further decomposition $L \cap U_i = \{p_i\} \cup (L_j - \{p_i, p_k\}) \cup (L_k - \{p_i, p_j\})$ into a total of nine pieces. Then a careful inspection of equation $F(X, \theta) = 0$ implies that around those pieces we can take local coordinates and local equations as in Table 2, where $O_m(u_j, u_k) = O((|u_j| + |u_k|)^m)$ denotes a small term of order m as $(u_j, u_k) \to (0, 0)$.

coordinates	valid around	local equation
(u_j, u_k)	p_i	$u_0 = -(u_j u_k)\{1 - (u_j^2 + \theta_i u_j u_k + u_k^2) + O_3(u_j, u_k)\}$
(u_0, u_k)	$L_j - \{p_i, p_k\}$	$u_j = -(u_k + 1/u_k)u_0 + (\theta_k + \theta_i/u_k)u_0^2 + O(u_0^3)$
(u_0, u_j)	$L_k - \{p_i, p_j\}$	$u_k = -(u_j + 1/u_j)u_0 + (\theta_j + \theta_i/u_j)u_0^2 + O(u_0^3)$
(v_i, v_k)	p_j	$v_0 = -(v_i v_k)\{1 - (v_i^2 + \theta_j v_i v_k + v_k^2) + O_3(v_i, v_k)\}$
(v_0, v_i)	$L_k - \{p_i, p_j\}$	$v_k = -(v_i + 1/v_i)v_0 + (\theta_i + \theta_j/v_i)v_0^2 + O(v_0^3)$
(v_0, v_k)	$L_i - \{p_j, p_k\}$	$v_i = -(v_k + 1/v_k)v_0 + (\theta_k + \theta_j/v_k)v_0^2 + O(v_0^3)$
(w_i, w_j)	p_k	$w_0 = -(w_i w_j)\{1 - (w_i^2 + \theta_k w_i w_j + w_j^2) + O_3(w_i, w_j)\}$
(w_0, w_j)	$L_i - \{p_j, p_k\}$	$w_i = -(w_j + 1/w_j)w_0 + (\theta_j + \theta_k/w_j)w_0^2 + O(w_0^3)$
(w_0, w_i)	$L_j - \{p_i, p_k\}$	$w_j = -(w_i + 1/w_i)w_0 + (\theta_i + \theta_k/w_i)w_0^2 + O(w_0^3)$

Table 2: Local coordinates and local equations of $\overline{\mathcal{S}}(\theta)$

Lemma 5.1 *As to the smoothness of the surface $\overline{\mathcal{S}}(\theta)$, the following hold.*

(1) *For any $\theta \in \Theta$, the surface $\overline{\mathcal{S}}(\theta)$ is smooth in a neighborhood of L.*

(2) *If $\theta = \mathrm{rh}(\kappa)$ with $\kappa \in \mathcal{K}$, the surface $\overline{\mathcal{S}}(\theta)$ is smooth everywhere if and only if $\kappa \in \mathcal{K}-$**Wall**.*

Proof. First we show assertion (1). In terms of inhomogeneous coordinates u in (20), we have

$$\overline{\mathcal{S}}(\theta) \cap U_i \cong \{\, u = (u_0, u_j, u_k) \in \mathbb{C}^3 \;:\; f_i(u, \theta) = 0 \,\},$$

where the defining equation $f_i(u, \theta)$ is given by

$$f_i(u, \theta) = u_j u_k + u_0(1 + u_j^2 + u_k^2) - u_0^2(\theta_i + \theta_j u_j + \theta_k u_k) + \theta_4 u_0^3.$$

The partial derivatives of $f_i = f_i(u, \theta)$ with respect to $u = (u_0, u_j, u_k)$ are calculated as

$$\begin{aligned}
\frac{\partial f_i}{\partial u_0} &= (1 + u_j^2 + u_k^2) - 2u_0(\theta_i + \theta_j u_j + \theta_k u_k) + 3\theta_4 u_0^2 \\
\frac{\partial f_i}{\partial u_j} &= u_k + 2u_0 u_j - \theta_j u_0^2 \\
\frac{\partial f_i}{\partial u_k} &= u_j + 2u_0 u_k - \theta_k u_0^2.
\end{aligned}$$

Restricted to $L \cap U_i = (L_j \cap U_i) \cup (L_k \cap U_i)$, these derivatives become

$$
\begin{aligned}
\frac{\partial f_i}{\partial u_0} &= 1 + u_k^2, & \frac{\partial f_i}{\partial u_j} &= u_k, & \frac{\partial f_i}{\partial u_k} &= 0, & &\text{on} \quad L_j \cap U_i, \\
\frac{\partial f_i}{\partial u_0} &= 1 + u_j^2, & \frac{\partial f_i}{\partial u_j} &= 0, & \frac{\partial f_i}{\partial u_k} &= u_j, & &\text{on} \quad L_k \cap U_i.
\end{aligned}
$$

Hence the exterior derivative $d_u f_i$ does not vanish on $L \cap U_i$, and the implicit function theorem implies that $\overline{\mathcal{S}}(\theta)$ is smooth in a neighborhood of L. This proves assertion (1). In order to show assertion (2) we recall that the affine surface $\mathcal{S}(\theta)$ is smooth if and only if $\theta = \mathrm{rh}(\kappa)$ with $\kappa \in \mathcal{K} - \mathbf{Wall}$ (see [13]). Then assertion (2) readily follows from assertion (1). $\square$

Now let us review some basic facts about smooth cubic surfaces in $\mathbb{P}^3$ (see e.g. [8]). It is well known that every smooth cubic surface S in $\mathbb{P}^3$ can be obtained by blowing up $\mathbb{P}^2$ at six points $P_1, \dots, P_6$, no three colinear and not all six on a conic, and embedding the blow-up surface into $\mathbb{P}^3$ by the proper transform of the linear system of cubics passing through the six points $P_1, \dots, P_6$. It is also well known that there are exactly 27 lines on the smooth cubic surface S, each of which has self-intersection number -1. Explicitly, they are given by

$$E_a \quad (a = 1, \dots, 6); \qquad F_{ab} \quad (1 \le a < b \le 6); \qquad G_a \quad (a = 1, \dots, 6),$$

(1) E_a is the exceptional curve over the point P_a,

(2) F_{ab} is the strict transform of the line in $\mathbb{P}^2$ through the two points P_a and P_b,

(3) G_a is the strict transform of the conic in $\mathbb{P}^2$ through the five points $P_1, \dots, \hat{P}_a, \dots, P_6$.

Here the index a should not be confused with the local monodromy data $a \in A$. All the intersection relations among the 27 lines with *nonzero* intersection numbers are listed as

$$
\begin{aligned}
(E_a, E_a) = (G_a, G_a) = (F_{ab}, F_{ab}) &= -1 & &(\forall\, a, b), \\
(E_a, F_{bc}) = (G_a, F_{bc}) &= 1 & &(a \in \{b, c\}), \\
(E_a, G_b) &= 1 & &(a \neq b), \\
(F_{ab}, F_{cd}) &= 1 & &(\{a, b\} \cap \{c, d\} = \emptyset).
\end{aligned}
$$

Moreover there are exactly 45 tritangent planes that cut out a triplet of lines on S. In our case $S = \overline{\mathcal{S}}(\theta)$, the plane at infinity $\{\, X \in \mathbb{P}^3 \,:\, X_0 = 0 \,\}$ is an instance of tritangent plane, which cuts out the lines in (19). Figure 7 offers an arrangement of the 27 lines viewed from the tritangent plane at infinity, where $\{i, j, k\} = \{1, 2, 3\}$ and $\{l, m, n\} = \{4, 5, 6\}$, and

$$L_i = F_{ij}, \qquad L_j = F_{kl}, \qquad L_k = F_{mn} \tag{21}$$

are allocated for the lines at infinity. Each line at infinity is intersected by exactly eight lines and this fact enables us to divide the 27 lines into three groups of nine lines labeled by lines at infinity. *Caution:* only the intersection relations among lines of the same group are indicated in Figure 7, with no other intersection relations being depicted.

If E_0 is the strict transform of a plane in $\mathbb{P}^2$ not passing through $P_1, \dots, P_6$ relative to the 6-point blow-up $S \to \mathbb{P}^2$, then the second cohomology group of $S = \overline{\mathcal{S}}(\theta)$ is expressed as

$$H^2(\overline{\mathcal{S}}(\theta), \mathbb{Z}) = \mathbb{Z}E_0 \oplus \mathbb{Z}E_i \oplus \mathbb{Z}E_j \oplus \mathbb{Z}E_k \oplus \mathbb{Z}E_l \oplus \mathbb{Z}E_m \oplus \mathbb{Z}E_n, \tag{22}$$

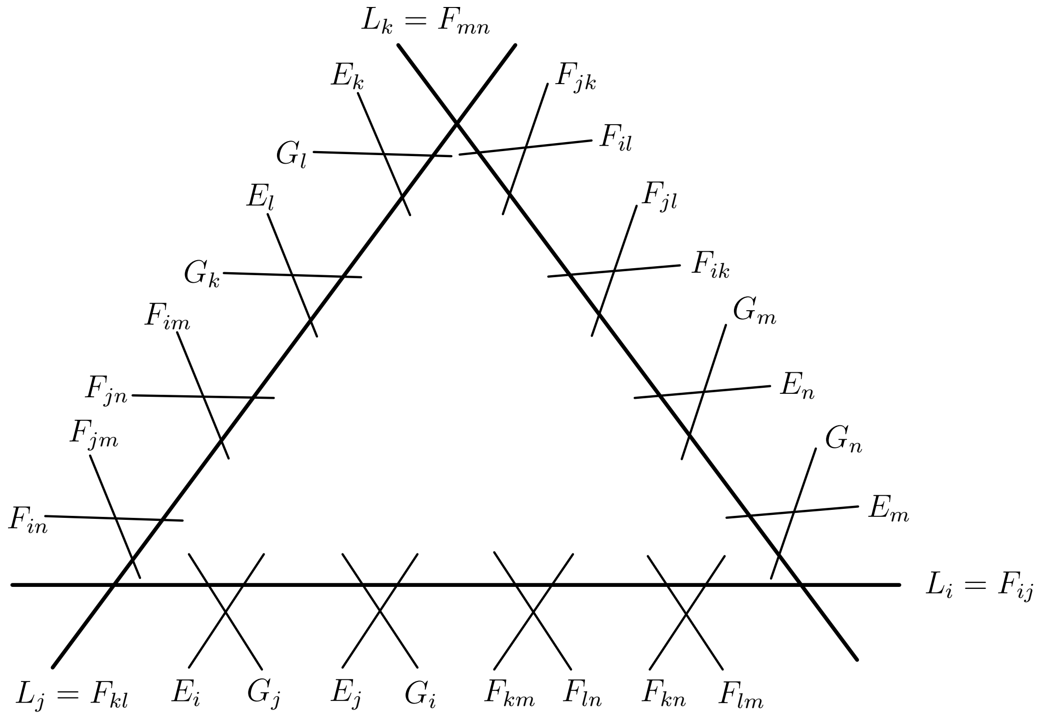

Figure 7: The 27 lines on $\overline{\mathcal{S}}(\theta)$ viewed from the tritangent plane at infinity

where a divisor is identified with the cohomology class it represents. It is a Lorentz lattice of rank 7 with intersection numbers

$$(E_a, E_b) = \begin{cases} 1 & (a = b = 0), \\ -1 & (a = b \neq 0), \\ 0 & (\text{otherwise}). \end{cases} \tag{23}$$

In terms of the basis in (22) the lines F_{ab} and G_a are represented as

$$F_{ab} = E_0 - E_a - E_b, \qquad G_a = 2E_0 - (E_1 + \cdots + \widehat{E}_a + \cdots + E_6). \tag{24}$$

We shall describe the 27 lines on our cubic surface $\overline{\mathcal{S}}(\theta)$ under the condition that $\overline{\mathcal{S}}(\theta)$ is smooth, namely, $\theta = \mathrm{rh}(\kappa)$ with $\kappa \in \mathcal{K} - \mathbf{Wall}$. To this end we introduce new parameters $b = (b_1, b_2, b_3, b_4) \in B := (\mathbb{C}_b^{\times})^4$ in such a manner that b is expressed as

$$b_i = \begin{cases} \exp(\sqrt{-1}\pi\kappa_i) & (i = 1, 2, 3), \\ -\exp(\sqrt{-1}\pi\kappa_4) & (i = 4), \end{cases}$$

as a function of $\kappa \in \mathcal{K}$. Then the Riemann scheme in Table 1 implies that b_i is an eigenvalue of the monodromy matrix $\rho(\gamma_i)$ around the point t_i and formula (14) implies that $a_i = b_i + b_i^{-1}$. Here parameters $b \in B$ should not be confused with the index b above. In terms of the parameters $b \in B$, the discriminant $\varDelta(\theta)$ of the cubic surfaces $\mathcal{S}(\theta)$ factors as

$$\varDelta(\theta) = \prod_{l=1}^{4} (b_l - b_l^{-1})^2 \prod_{\varepsilon \in \{\pm 1\}^4} (b^{\varepsilon} - 1), \tag{25}$$

1	$L_i(b_i, b_4; b_j, b_k)$	$L_i(1/b_i, 1/b_4; b_j, b_k)$
2	$L_i(b_j, b_k; b_i, b_4)$	$L_i(1/b_j, 1/b_k; b_i, b_4)$
3	$L_i(1/b_i, b_4; b_j, b_k)$	$L_i(b_i, 1/b_4; b_j, b_k)$
4	$L_i(1/b_j, b_k; b_i, b_4)$	$L_i(b_j, 1/b_k; b_i, b_4)$

Table 3: Eight lines intersecting the line L_i at infinity, divided into four pairs

where $b^{\varepsilon} = b_1^{\varepsilon_1} b_2^{\varepsilon_2} b_3^{\varepsilon_3} b_4^{\varepsilon_4}$ for each quadruple sign $\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) \in \{\pm 1\}^4$. Formula (25) clearly shows for which parameters $b \in B$ the cubic surface $\overline{\mathcal{S}}(\theta)$ is smooth or singular.

Let $L_i(b_i, b_4; b_j, b_k)$ denote the line in $\mathbb{P}^3$ defined by the system of linear equations

$$X_i = (b_i b_4 + b_i^{-1} b_4^{-1}) X_0, \qquad X_j + (b_i b_4) X_k = \{b_i(b_k + b_k^{-1}) + b_4(b_j + b_j^{-1})\} X_0. \tag{26}$$

Assume that $\overline{\mathcal{S}}(\theta)$ is smooth, namely, $\mathit{\Delta}(\theta) \neq 0$. Then, as is mentioned earlier, for each $i \in \{1, 2, 3\}$ there are exactly eight lines on $\overline{\mathcal{S}}(\theta)$ intersecting L_i, but not intersecting the remaining two lines at infinity, L_j and L_k. They are just $\{E_i, G_j\}$, $\{E_j, G_i\}$, $\{F_{km}, F_{ln}\}$, $\{F_{kn}, F_{lm}\}$ as in Figure 7, where two lines from the same pair intersect, but ones from different pairs are disjoint. In terms of parameters $b \in B$ those eight lines are given as in Table 3.

6 Involutions on Cubic Surface

The affine cubic surface $\mathcal{S}(\theta)$ is a $(2, 2, 2)$-surface, that is, its defining equation $f(x, \theta) = 0$ is a quadratic equation in each variable x_i, $i = 1, 2, 3$. Therefore the line through a point $x \in \mathcal{S}(\theta)$ parallel to the x_i-axis passes through a unique second point $x' \in \mathcal{S}(\theta)$ (see Figure 8). This defines an involution $\sigma_i : \mathcal{S}(\theta) \to \mathcal{S}(\theta)$, $x \mapsto x'$, which is explicitly given by

$$\sigma_i : \qquad (x_i', x_j', x_k') = (\theta_i - x_i - x_j x_k, x_j, x_k), \qquad (i = 1, 2, 3). \tag{27}$$

The automorphism σ_i of the affine surface $\mathcal{S}(\theta)$ extends to a birational map of the projective surface $\overline{\mathcal{S}}(\theta)$, which will also be denoted by σ_i. In terms of the homogeneous coordinates X of $\mathbb{P}^3$, the birational map $\sigma_i : X \mapsto X'$ is expressed as

$$[X_0' : X_i' : X_j' : X_k'] = [X_0^2 : \theta_i X_0^2 - X_0 X_i - X_j X_k : X_0 X_j : X_0 X_k]$$

We shall investigate the behavior of the birational map σ_i in a neighborhood of the tritangent lines L at infinity. To this end let us introduce the following three points

$$q_1 = [0 : 0 : 1 : 1], \qquad q_2 = [0 : 1 : 0 : 1], \qquad q_3 = [0 : 1 : 1 : 0],$$

where q_i may be thought of as the "mid-point" of p_j and p_k on the line L_i.

Lemma 6.1 *The birational map σ_i has the following properties (see Figure 9).*

(1) *σ_i blows down the line L_i to the point p_i,*

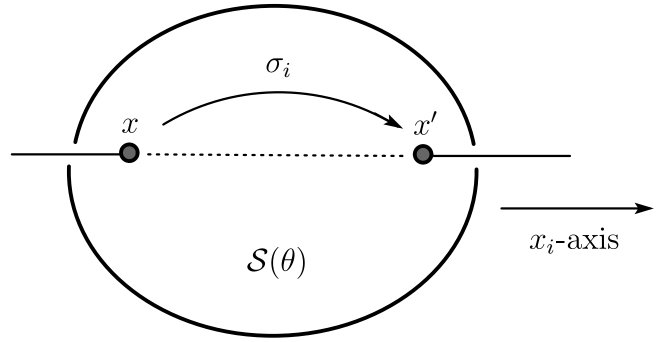

Figure 8: Involutions of a $(2,2,2)$-surface

(2) *σ_i restricts to the automorphism of L_j that fixes q_j and exchanges p_i and p_k,*

(3) *σ_i restricts to the automorphism of L_k that fixes q_k and exchanges p_i and p_j,*

(4) *p_i is the unique indeterminacy point of σ_i,*

Proof. In order to investigate σ_i, we make use of inhomogeneous coordinates of $\mathbb{P}^3$ in (20) and local coordinates and local equations of $\overline{\mathcal{S}}(\theta)$ in Table 2, with target coordinates being dashed.

In terms of inhomogeneous coordinates v and u' of $\mathbb{P}^3$, the map $\sigma_i : v \mapsto u'$ is expressed as

$$u'_0 = \frac{v_0^2}{\theta_i v_0^2 - v_0 v_i - v_k}, \qquad u'_j = \frac{v_0}{\theta_i v_0^2 - v_0 v_i - v_k}, \qquad u'_k = \frac{v_0 v_k}{\theta_i v_0^2 - v_0 v_i - v_k}. \tag{28}$$

In a neighborhood of $L_i - \{p_j, p_k\}$ in $\overline{\mathcal{S}}(\theta)$, using $v_i = O(v_0)$, we observe that

$$\theta_i v_0^2 - v_0 v_i - v_k = -v_k\{1 + O(v_0^2)\},$$

which is substituted into (28) to yield

$$u'_j = -\frac{v_0}{v_k\{1 + O(v_0^2)\}} = -\frac{v_0}{v_k}\{1 + O(v_0^2)\}, \qquad u'_k = -\frac{v_0 v_k}{v_k\{1 + O(v_0^2)\}} = -v_0\{1 + O(v_0^2)\}.$$

In particular putting $v_0 = 0$ leads to $u'_j = u'_k = 0$. This means that σ_i maps a neighborhood of $L_i - \{p_j, p_k\}$ to a neighborhood of p_i, collapsing $L_i - \{p_j, p_k\}$ to the single point p_i.

In a similar manner, in a neighborhood of p_j in $\overline{\mathcal{S}}(\theta)$ we observe that

$$v_0 = -(v_i v_k)\{1 + O_2(v_i, v_k)\}, \qquad \theta_i v_0^2 - v_0 v_i - v_k = -v_k\{1 + O_2(v_i, v_k)\},$$

which are substituted into (28) to yield

$$u'_j = v_i\{1 + O_2(v_i, v_k)\}, \qquad u'_k = (v_i v_k)\{1 + O_2(v_i, v_k)\}.$$

In particular putting $v_i = 0$ leads to $u'_j = u'_k = 0$. This means that σ_i maps a neighborhood of p_j to a neighborhood of p_i, collapsing a neighborhood in L_i of p_j to the single point p_i. Using w and u' in place of v and u', we can make a similar argument in a neighborhood of p_k. Therefore σ_i blows down L_i to the point p_i, which proves assertion (1). Moreover it is clear from the argument that there is no indeterminacy point on the line L_i.

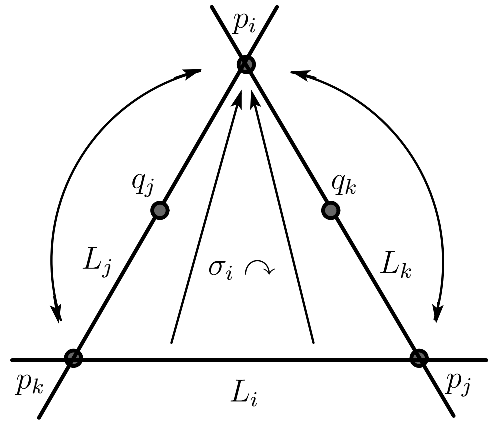

Figure 9: The birational map σ_i restricted to L

In terms of inhomogeneous coordinates u and u' of $\mathbb{P}^3$ the map $\sigma_i : u \mapsto u'$ is expressed as

$$u_0' = \frac{u_0^2}{\theta_i u_0^2 - u_0 - u_j u_k}, \quad u_j' = \frac{u_0 u_j}{\theta_i u_0^2 - u_0 - u_j u_k}, \quad u_k' = \frac{u_0 u_k}{\theta_i u_0^2 - u_0 - u_j u_k}. \tag{29}$$

In a neighborhood of $L_j - \{p_i, p_k\}$ in $\overline{\mathcal{S}}(\theta)$, using $u_j = -(u_k + 1/u_k)u_0 + O(u_0^2)$, we have

$$\theta_i u_0^2 - u_0 - u_j u_k = u_0\{u_k^2 + O(u_0)\},$$

which is substituted into (29) to yield

$$u_0' = \frac{u_0}{u_k^2 + O(u_0)} = \frac{u_0}{u_k^2} + O(u_0^2), \qquad u_k' = \frac{u_k}{u_k^2 + O(u_0)} = \frac{1}{u_k} + O(u_0).$$

In particular putting $u_0 = 0$ leads to $u_0' = 0$ and $u_k' = 1/u_k$. This means that σ_i restricts to an automorphism of a neighborhood of $L_j - \{p_i, p_k\}$ in $\overline{\mathcal{S}}(\theta)$ that induces a unique automorphism of L_i fixing q_j and exchanging p_i and p_k. This proves assertion (2) and also shows that there is no indeterminacy point on $L_j - \{p_i, p_k\}$. Assertion (3) and the nonexistence of indeterminacy point on $L_k - \{p_i, p_j\}$ are established just in the same manner.

From the above argument we have already known that there is no indeterminacy point other than p_i. Then the point p_i is actually an indeterminacy point, because σ_i is an involution blowing down L_i to p_i and hence blows up p_i to L_i reciprocally. This proves assertion (4). $\square$

Later we will need some information about how the involution σ_i transforms a line to another curve, which is stated in the following lemma.

Lemma 6.2 *The involution σ_i satisfies the following properties:*

(1) *$\sigma_i(E_i)$ intersects E_i at two points counted with multiplicity,*

(2) *$\sigma_i(E_i)$ intersects E_j at one point counted with multiplicity,*

(3) *σ_i exchanges the lines E_k and G_l; E_l and G_k; E_m and G_n; E_n and G_m, respectively.*

Proof. By Table 3 we may put $E_i = L_i(b_i, b_4; b_j, b_k)$ and $E_j = L_i(b_j, b_k; b_i, b_4)$. Assertion (1) of Lemma 6.1 implies that $\sigma_i(E_i)$ does not intersect E_i nor E_j at any point at infinity. So we can work with inhomogeneous coordinates $x = (x_1, x_2, x_3)$. In view of (26) the line E_i is given by

$$x_i = b_i b_4 + (b_i b_4)^{-1}, \qquad x_j + (b_i b_4) x_k = a_k b_i + a_j b_4. \tag{30}$$

In a similar manner, by exchanging (b_i, b_4) and (b_j, b_k) in (26), the line E_j is given by

$$x_i = b_j b_k + (b_j b_k)^{-1}, \qquad x_j + (b_j b_k) x_k = a_4 b_j + a_i b_k. \tag{31}$$

Moreover, by applying formula (27) to (30), the curve $\sigma_i(E_i)$ is expressed as

$$\theta_i - x_i - x_j x_k = b_i b_4 + (b_i b_4)^{-1}, \qquad x_j + (b_i b_4) x_k = a_k b_i + a_j b_4. \tag{32}$$

Note that the second equations of (30) and (32) are the same.

In order to find out the intersection of $\sigma_i(E_i)$ with E_i, let us couple (30) and (32). Eliminating x_i and x_j we obtain a quadratic equation for x_k,

$$(b_i b_4) x_k^2 - (a_k b_i + a_j b_4) x_k + \theta_i - 2\{b_i b_4 + (b_i b_4)^{-1}\} = 0.$$

For each root of this equation we have an intersection point of $\sigma_i(E_i)$ with E_i; for a double root we have an intersection point of multiplicity two. This proves assertion (1).

Next, in order to find out the intersection of $\sigma_i(E_i)$ with E_j, let us couple (31) and (32). From the first equation of (31) the x_i-coordinate is already fixed. The second equations of (31) and (32) are coupled to yield a linear system for x_j and x_k, whose determinant

$$b_j b_k - b_i b_4 = b_i b_4 (b_i^{-1} b_j b_k b_4^{-1} - 1)$$

is nonzero from the assumption that $\overline{\mathcal{S}}(\theta)$ is smooth, that is, the discriminant $\varDelta(\theta)$ in (25) is nonzero. Then the linear system is uniquely solved to determine x_j and x_k. Now we can check that the first equation of (32) is redundant, that is, automatically satisfied. Therefore $\sigma_i(E_i)$ and E_j has a simple intersection, which implies assertion (2).

Finally we see that σ_i exchanges E_k and G_l. We may put $E_k = L_j(b_j, b_4; b_k, b_i)$ and $G_l = L_j(1/b_j, 1/b_4; b_k, b_i)$. By formula (26) (with indices suitably permuted), these lines are given by

$$x_j = b_j b_4 + (b_j b_4)^{-1}, \qquad x_k + (b_j b_4) x_i = a_i b_j + a_k b_4, \tag{33}$$

$$x_j = b_j b_4 + (b_j b_4)^{-1}, \qquad x_k + (b_j b_4)^{-1} x_i = a_i b_j^{-1} + a_k b_4^{-1}. \tag{34}$$

Using formula (27) we can check that equations (33) and (34) are transformed to each other by σ_i. This together with similar argument for the other lines establishes assertion (3). □

7 Dynamical System on Cubic Surface

Let $G = \langle \sigma_1, \sigma_2, \sigma_3 \rangle$ be the group of birational transformations on $\overline{\mathcal{S}}(\theta)$ generated by the involutions σ_1, σ_2, σ_3. We are interested in the dynamics of the G-action on $\overline{\mathcal{S}}(\theta)$. Usually the dynamics of a group action is more involved than that of a single transformation; more techniques and tools have been developed for the latter rather than for the former. So it may

be better to pick up a single transformation from the group G and study its dynamics. For such a transformation we take a composition of the three basic involutions,

$$c = \sigma_i \circ \sigma_j \circ \sigma_k \, : \, \overline{\mathcal{S}}(\theta) \circlearrowleft . \tag{35}$$

If G is regarded as a nonlinear reflection group with basic 'reflections' σ_1, σ_2, σ_3, then c may be thought of as a 'Coxeter' transformation and it is expected that the dynamics of the transformation c plays a dominant role in understanding the dynamics of the whole G-action.

The relevance of the transformation (35) to our main problem is stated as follows.

Lemma 7.1 *Via the Riemann-Hilbert correspondence (17) the Pochhammer-Poincaré map* $\wp_* : M_x(\kappa) \circlearrowleft$ *is strictly conjugated to the square* $c^2 : \mathcal{S}(\theta) \circlearrowleft$ *of the Coxeter transformation (35), restricted to the affine part* $\mathcal{S}(\theta)$ *of the cubic surface* $\overline{\mathcal{S}}(\theta)$.

Proof. Since the transformation g_i in (18) is a strict conjugacy of the half-Poincaré map β_{i*} in (12), a glance at (11) and (12) shows that the commutator $[g_i^2, g_j^{-2}] = g_i^2 g_j^{-2} g_i^{-2} g_j^2$ is a strict conjugacy of the Pochhammer-Poincaré map $\wp_*$. On the other hand, using (18) and (27), we can directly check that $g_i^2 g_j^{-2} g_i^{-2} g_j^2 = (\sigma_i \sigma_j \sigma_k)^2 = c^2$. Hence c^2 is a strict conjugacy of $\wp_*$. $\square$

A general theory of dynamical systems for bimeromorphic maps of surfaces is developed in [5]. We shall apply it to our map (35) upon reviewing some rudiments of the article [5]. Let S be a compact complex surface, $f : S \circlearrowleft$ a bimeromorphic map. Then f is represented by a compact complex surface Γ and proper modifications $\pi_1 : \Gamma \to S$ and $\pi_2 : \Gamma \to S$ such that $f = \pi_2 \circ \pi_1^{-1}$ on a dense open subset. For $i = 1, 2$, let $\mathcal{E}(\pi_i) := \{ x \in \Gamma \, : \, \# \pi_i^{-1}(\pi_i(x)) = \infty \}$ be the exceptional set for the projection π_i. The images $I(f) := \pi_1(\mathcal{E}(\pi_1))$ and $\mathcal{E}(f) := \pi_1(\mathcal{E}(\pi_2))$ are called the indeterminacy set and the exceptional set of f respectively. In our case where $S = \overline{\mathcal{S}}(\theta)$ and $f = \sigma_i$, Lemma 6.1 implies that these sets are described as follows.

Lemma 7.2 $I(\sigma_i) = \{p_i\}$ *and* $\mathcal{E}(\sigma_i) = L_i$ *for* $i = 1, 2, 3$.

If S is Kähler, two natural actions of f, pull-back and push-forward, on the Dolbeault cohomology group $H^{1,1}(S)$ are defined in the following manner: A smooth $(1,1)$-form ω on S can be pulled back as a smooth $(1,1)$-form $\pi_2^*\omega$ on Γ and then pushed forward as a $(1,1)$-current $\pi_{1*}\pi_2^*\omega$ on S. Hence we define the pull-back $f^*\omega := \pi_{1*}\pi_2^*\omega$ and also the push-forward $f_*\omega = (f^{-1})^*\omega := \pi_{2*}\pi_1^*\omega$. The operators f^* and f_* commute with the exterior differential d and the complex structure of S and so descend to linear actions on $H^{1,1}(S)$. For general bimeromorphic maps f and g, the composition rule $(f \circ g)^* = g^* \circ f^*$ is not necessarily true. But a useful criterion under which this rule becomes true is given in [5].

Lemma 7.3 *If* $f(\mathcal{E}(f)) \cap I(g) = \emptyset$, *then* $(f \circ g)^* = g^* \circ f^* : H^{1,1}(S) \circlearrowleft$.

We apply this lemma to our Coxeter transformation $c = \sigma_i \circ \sigma_j \circ \sigma_k$.

Lemma 7.4 *We have* $c^* = \sigma_k^* \circ \sigma_j^* \circ \sigma_i^* : H^{1,1}(\overline{\mathcal{S}}(\theta)) \circlearrowleft$.

Proof. First we apply Lemma 7.3 to $f = \sigma_i$ and $g = \sigma_j \circ \sigma_k$. By Lemmas 6.1 and 7.2 we have $\mathcal{E}(\sigma_i) = L_i$ and $I(\sigma_j \circ \sigma_k) = \{p_k\}$ and so $\sigma_i(\mathcal{E}(\sigma_i)) \cap I(\sigma_j \circ \sigma_k) = \{p_i\} \cap \{p_k\} = \emptyset$, which means that the condition of Lemma 7.3 is satisfied. Then the lemma yields $(\sigma_i \circ \sigma_j \circ \sigma_k)^* = (\sigma_j \circ \sigma_k)^* \circ \sigma_i^*$. Next we apply Lemma 7.3 to $f = \sigma_j$ and $g = \sigma_k$. Again by Lemmas 6.1 and 7.2 we have $\mathcal{E}(\sigma_j) = L_j$ and $I(\sigma_k) = \{p_k\}$ and so $\sigma_j(\mathcal{E}(\sigma_j)) \cap I(\sigma_k) = \{p_j\} \cap \{p_k\} = \emptyset$, which means that

$$\sigma_i^* = \begin{pmatrix} 6 & 3 & 3 & 2 & 2 & 2 & 2 \\ -3 & -2 & -1 & -1 & -1 & -1 & -1 \\ -3 & -1 & -2 & -1 & -1 & -1 & -1 \\ -2 & -1 & -1 & -1 & 0 & -1 & -1 \\ -2 & -1 & -1 & 0 & -1 & -1 & -1 \\ -2 & -1 & -1 & -1 & -1 & -1 & 0 \\ -2 & -1 & -1 & -1 & -1 & 0 & -1 \end{pmatrix} \qquad \sigma_j^* = \begin{pmatrix} 6 & 2 & 2 & 3 & 3 & 2 & 2 \\ -2 & -1 & 0 & -1 & -1 & -1 & -1 \\ -2 & 0 & -1 & -1 & -1 & -1 & -1 \\ -3 & -1 & -1 & -2 & -1 & -1 & -1 \\ -3 & -1 & -1 & -1 & -2 & -1 & -1 \\ -2 & -1 & -1 & -1 & -1 & -1 & 0 \\ -2 & -1 & -1 & -1 & -1 & 0 & -1 \end{pmatrix}$$

$$\sigma_k^* = \begin{pmatrix} 6 & 2 & 2 & 2 & 2 & 3 & 3 \\ -2 & -1 & 0 & -1 & -1 & -1 & -1 \\ -2 & 0 & -1 & -1 & -1 & -1 & -1 \\ -2 & -1 & -1 & -1 & 0 & -1 & -1 \\ -2 & -1 & -1 & 0 & -1 & -1 & -1 \\ -3 & -1 & -1 & -1 & -1 & -2 & -1 \\ -3 & -1 & -1 & -1 & -1 & -1 & -2 \end{pmatrix} \qquad c^* = \begin{pmatrix} 12 & 6 & 6 & 4 & 4 & 3 & 3 \\ -3 & -2 & -1 & -1 & -1 & -1 & -1 \\ -3 & -1 & -2 & -1 & -1 & -1 & -1 \\ -4 & -2 & -2 & -2 & -1 & -1 & -1 \\ -4 & -2 & -2 & -1 & -2 & -1 & -1 \\ -6 & -3 & -3 & -2 & -2 & -2 & -1 \\ -6 & -3 & -3 & -2 & -2 & -1 & -2 \end{pmatrix}$$

Table 4: Matrix representations of σ_i^*, σ_j^*, σ_k^*, $c^* : H^2(\overline{\mathcal{S}}(\theta), \mathbb{Z}) \circlearrowleft$

the condition of Lemma 7.3 is satisfied. Then the lemma yields $(\sigma_j \circ \sigma_k)^* = \sigma_k^* \circ \sigma_j^*$. Putting these two steps together, we obtain $c^* = (\sigma_i \circ \sigma_j \circ \sigma_k)^* = (\sigma_j \circ \sigma_k)^* \circ \sigma_i^* = \sigma_k^* \circ \sigma_j^* \circ \sigma_i^*$. $\square$

By Lemma 7.4 the calculation of the action $c^* : H^{1,1}(\overline{\mathcal{S}}(\theta)) \circlearrowleft$ is reduced to that of the actions σ_i^*, σ_j^*, σ_k^* : $H^{1,1}(\overline{\mathcal{S}}(\theta)) \circlearrowleft$, which is now set forth. Since the cubic surface $\overline{\mathcal{S}}(\theta)$ is rational, we have $H^{1,1}(\overline{\mathcal{S}}(\theta)) = H^2(\overline{\mathcal{S}}(\theta), \mathbb{C})$, where the latter group is described in (22).

Proposition 7.5 *The linear maps σ_i^*, σ_j^*, σ_k^*, $c^* : H^2(\overline{\mathcal{S}}(\theta), \mathbb{Z}) \circlearrowleft$ admit matrix representations as in Table 4 with respect to the basis in (22). The characteristic polynomial of c^* is given by*

$$\det(xI - c^*) = x(x+1)^4(x^2 - 4x - 1), \tag{36}$$

and hence its eigenvalues are 0, -1 and $2 \pm \sqrt{5}$, where the eigenvalue -1 is quadruple while the remaining ones are all simple. The spectral radius $\rho(c^*)$ of c^* is given by $2 + \sqrt{5}$.

Proof. First we shall find the matrix representation of σ_i^*. If ξ_{ab} denotes the (a,b)-th entry of the matrix to be found, then (23) implies that

$$\sigma_i^* E_b = \sum_{a=0}^{6} \xi_{ab}\, E_a = \sum_{a=0}^{6} \delta_a(\sigma_i^* E_b, E_a)\, E_a,$$

where we put $\delta_a = 1$ for $a = 0$ and $\delta_a = -1$ for $a \neq 0$. Now we claim that

$$\xi_{ab} = \delta_a(\sigma_i^* E_b, E_a), \qquad \xi_{ab} = \delta_a \delta_b \xi_{ba}. \tag{37}$$

The first formula in (37) is obvious and the second formula is derived as follows:

$$\xi_{ab} = \delta_a(\sigma_i^* E_b, E_a) = \delta_a(E_b, \sigma_{i*} E_a) = \delta_a(E_b, \sigma_i^* E_a) = (\delta_a\delta_b) \cdot \delta_b(\sigma_i^* E_a, E_b) = (\delta_a\delta_b)\xi_{ba},$$

where in the third equality we have used the fact that σ_i is an involution; $\sigma_{i*} = (\sigma_i^{-1})^* = \sigma_i^*$. By assertions (1) and (2) of Lemma 6.2 we have $(\sigma_i^* E_i, E_i) = 2$ and $(\sigma_i^* E_i, E_j) = 1$ and likewise $(\sigma_i^* E_j, E_j) = 2$ and $(\sigma_i^* E_j, E_i) = 1$. Then the first formula of (37) yields

$$\xi_{ii} = \xi_{jj} = -2, \qquad \xi_{ij} = \xi_{ji} = -1. \tag{38}$$

The assertion (3) of Lemma 6.2 together with the second formula of (24) yields

$$\begin{cases} \sigma_i^* E_k &= 2E_0 - E_i - E_j - E_k \qquad\quad - E_m - E_n, \\ \sigma_i^* E_l &= 2E_0 - E_i - E_j \qquad\quad - E_l - E_m - E_n, \\ \sigma_i^* E_m &= 2E_0 - E_i - E_j - E_k - E_l - E_m \qquad\quad , \\ \sigma_i^* E_n &= 2E_0 - E_i - E_j - E_k - E_l \qquad\quad - E_n, \end{cases} \tag{39}$$

It follows from (38) and (39) that the matrix representation for σ_i^* takes the form

$$\sigma_i^* = \left(\begin{array}{ccc|cccc} * & * & * & 2 & 2 & 2 & 2 \\ * & -2 & -1 & -1 & -1 & -1 & -1 \\ * & -1 & -2 & -1 & -1 & -1 & -1 \\ \hline * & * & * & -1 & 0 & -1 & -1 \\ * & * & * & 0 & -1 & -1 & -1 \\ * & * & * & -1 & -1 & -1 & 0 \\ * & * & * & -1 & -1 & 0 & -1 \end{array}\right), \tag{40}$$

where the entries denoted by $*$ are yet to be determined. The $(2,1)$-block of (40) is easily determined by the second formula in (37). The final ingredient taken into account is the fact that σ_i blows down $L_i = E_0 - E_i - E_j$ to a point p_i (see Lemma 6.1), which leads to

$$\sigma_i^* E_0 - \sigma_i^* E_i - \sigma_i^* E_j = 0.$$

This means that the first column is the sum of the second and third columns in the matrix (40). Using the second formula in (37) repeatedly, we see that the matrix representation of σ_i^* is given as in the first matrix of Table 4. Those of σ_j^* and σ_k^* are obtained in the same manner. Applying Lemma 7.4 to these results yields the desired representation for c^* as in the last matrix of Table 4. Now it is easy to calculate the characteristic polynomial of c^* as in (36). The assertion for its roots, namely, for the eigenvalues of c^* is straightforward. $\square$

We recall some more rudiments from [5]. Given a bimeromorphic map f of a compact Kähler surface S, there is the concept of *first dynamical degree* $\lambda_1(f)$ defined by

$$\lambda_1(f) := \lim_{n\to\infty} ||(f^n)^*||^{1/n},$$

where $||\cdot||$ is an operator norm on End $H^{1,1}(S)$. The limit certainly exists and one has $\lambda_1(f) \geq 1$. It is usually difficult to evaluate this number in a simple mean. But there is a distinguished class of bimeromorphic maps for which the first dynamical degree can be equated to a more tractable quantity, namely, the class of maps which are called analytically stable. Here a bimeromorphic map $f : S \circlearrowleft$ is said to be *analytically stable* (AS for short) if for any $n \in \mathbb{N}$ there is no curve $V \subset S$ such that $f^n(V) \subset I(f)$. From [5] we have the following lemma.

Lemma 7.6 *If $f : S \circlearrowleft$ is an AS bimeromorphic map, then the first dynamical degree $\lambda_1(f)$ is equal to the spectral radius $\rho(f^*)$ of the linear map $f^* : H^{1,1}(S) \circlearrowleft$.*

With this lemma in hand we continue to investigate the Coxeter transformation (35).

Proposition 7.7 *The birational map $c = \sigma_i \circ \sigma_j \circ \sigma_k$ enjoys the following properties:*

(1) *its indeterminacy set is given by $I(c) = \{p_k\}$,*

(2) *its exceptional set is given by $\mathcal{E}(c) = L$ with image $c(\mathcal{E}(c)) = \{p_i\}$,*

(3) *its tangent map $(dc)_{p_i}$ at p_i is zero, that is, p_i is a superattracting fixed point,*

(4) *it is AS, and*

(5) *its first dynamical degree is given by $\lambda_1(c) = 2 + \sqrt{5}$.*

Proof. Lemma 6.1 implies that $\sigma_k^{-1}(I(\sigma_j)) = \{p_k\}$ and $\sigma_k^{-1} \circ \sigma_j^{-1}(I(\sigma_i)) = \sigma_k^{-1}(\{p_j\}) = \{p_k\}$. Thus the indeterminacy set of c is given by $I(c) = \{p_k\}$, which proves assertion (1). In order to see assertion (2), we again apply Lemma 6.1 to obtain

$$\begin{array}{rcccccll} c(L_i) & = & \sigma_i \circ \sigma_j \circ \sigma_k(L_i) & = & \sigma_i \circ \sigma_j(L_i) & = & \sigma_i(L_i) & = \{p_i\}, \\ c(L_j) & = & \sigma_i \circ \sigma_j \circ \sigma_k(L_j) & = & \sigma_i \circ \sigma_j(L_j) & = & \sigma_i(\{p_j\}) & = \{p_i\}, \\ c(L_k) & = & \sigma_i \circ \sigma_j \circ \sigma_k(L_k) & = & \sigma_i \circ \sigma_j(\{p_k\}) & = & \sigma_i(\{p_j\}) & = \{p_i\}. \end{array}$$

This means that $\mathcal{E}(c)$ is given by the union $L = L_i \cup L_j \cup L_k$ with $c(\mathcal{E}(c)) = \{p_i\}$. Thus assertion (2) follows. From assertion (2) we notice that p_i is a fixed point of c and all points on $L_j - \{p_k\}$ and on $L_k - \{p_j\}$ are taken to the point p_i by c. So the tangent map $(dc)_{p_i}$ is zero along the linearly independent directions of the lines L_j and L_k with origin at p_i and hence $(dc)_{p_i}$ itself is zero, which proves assertion (3). We show assertion (4) by contradiction. Assume that $V \subset \overline{\mathcal{S}}(\theta)$ is an irreducible curve such that $c^n(V) \subset I(c) = \{p_k\}$ for some $n \in \mathbb{N}$. If V intersects the affine surface $\mathcal{S}(\theta)$, then it cannot happen that $c^n(V) \subset \{p_k\}$, because c is bijective on $\mathcal{S}(\theta)$. Thus V must lie in L and hence $V = L_i$, L_j, or L_k. But also in this case assertion (2) implies that $c^n(L_a) = \{p_k\}$ for $a = i, j, k$, leading to a contradiction. Hence assertion (4) is proved. Finally, since c is AS, Proposition 7.5 and Lemma 7.6 immediately imply assertion (5). $\square$

We conjecture that the topological entropy of c agrees with the logarithm of its first dynamical degree:

$$h_{\mathrm{top}}(c) = \log \lambda_1(c) = \log(2 + \sqrt{5}).$$

8 Lefschetz Fixed Point Formula

We are interested in the periodic points of the Coxeter transformation $c : \overline{\mathcal{S}}(\theta) \circlearrowleft$. Given any $N \in \mathbb{N}$ we can consider the set of periodic points of period N on the projective surface $\overline{\mathcal{S}}(\theta)$,

$$\overline{\mathrm{Per}}_N(c) := \{\, X \in \overline{\mathcal{S}}(\theta) - I(c^N) \;:\; c^N(X) = X \,\}.$$

as well as the set of periodic points of period N on the affine surface $\mathcal{S}(\theta)$,

$$\mathrm{Per}_N(c) := \{\, x \in \mathcal{S}(\theta) \;:\; c^N(x) = x \,\},$$

On the other hand we have defined in (5) the set $\mathrm{Per}_N(\kappa)$ of periodic points of period N for the Pochhammer-Poincaré map $\wp_*$. By Lemma 7.1, $\mathrm{Per}_N(\kappa)$ is bijectively mapped onto $\mathrm{Per}_N(c^2) = \mathrm{Per}_{2N}(c)$ by the Riemann-Hilbert correspondence (17) and hence

$$\# \mathrm{Per}_N(\kappa) = \# \mathrm{Per}_{2N}(c). \tag{41}$$

Thus the main aim of this article, that is, the enumeration of the set $\mathrm{Per}_N(\kappa)$ is reduced to that of $\mathrm{Per}_N(c)$. So what we should do from now on is the following:

- to count the cardinality of $\overline{\mathrm{Per}}_N(c)$,
- to relate the cardinality of $\overline{\mathrm{Per}}_N(c)$ with that of $\mathrm{Per}_N(c)$.

The first task will be done with the help of Lefschetz fixed point formula and the second task will be by a careful inspection of the behavior of c around L. In order to apply the Lefschetz fixed point formula, we first need to verify the following lemma.

Lemma 8.1 *For any $N \in \mathbb{N}$, the Coxeter transformation $c : \overline{\mathcal{S}}(\theta) \circlearrowleft$ admits no curves of periodic points of period N.*

Proof. By Proposition 7.5 the Coxeter transformation $c^* : H^2(\overline{\mathcal{S}}(\theta), \mathbb{Z}) \circlearrowleft$ has eigenvalues 0, -1, $2 \pm \sqrt{5}$, among which 0 and $2 \pm \sqrt{5}$ are simple eigenvalues, while -1 is a quadruple eigenvalue whose eigenspace is spanned by four eigenvectors

$$V_0 = 2E_0 - E_i - E_j - E_k - E_l - E_m - E_n, \quad V_i = E_i - E_j, \quad V_j = E_k - E_l, \quad V_k = E_m - E_n.$$

In view of (21), (23) and (24), there are orthogonality relations

$$(V_a, L_b) = 0 \qquad (a = 0, i, j, k, \; b = i, j, k). \tag{42}$$

We prove the lemma by contradiction. Assume that c admits a curve (an effective divisor) $D \subset \overline{\mathcal{S}}(\theta)$ of periodic points of some period N. Then we have $(c^*)^N D = (c^N)^* D = D$ in $H^2(\overline{\mathcal{S}}(\theta), \mathbb{Z})$, where $(c^*)^N = (c^N)^*$ follows from the fact that c is AS. So $(c^*)^N$ has eigenvalue 1 with eigenvector D. This eigenvalue arises as the N-th power of eigenvalue -1 of c^* so that N must be even and D must be a linear combination of V_0, V_i, V_j, V_k. Hence (42) implies that

$$(D, L_a) = 0 \qquad (a = i, j, k). \tag{43}$$

We now write $D = D' + m_i L_i + m_j L_j + m_k L_k$, where D' is either empty or an effective divisor not containing L_i, L_j, L_k as an irreducible component of it and m_i, m_j, m_k are nonnegative integers. Since $(L_a, L_b) = -1$ for $a = b$ and $(L_a, L_b) = 1$ for $a \neq b$, the formula (43) yields

$$\begin{aligned} 0 &= (D, L_i) &= (D', L_i) - m_i + m_j + m_k, \\ 0 &= (D, L_j) &= (D', L_j) + m_i - m_j + m_k, \\ 0 &= (D, L_k) &= (D', L_k) + m_i + m_j - m_k, \end{aligned}$$

which sum up to

$$(D', L_i) + (D', L_j) + (D', L_k) + m_i + m_j + m_k = 0. \tag{44}$$

Since none of L_i, L_j, L_k is an irreducible component of D', the intersection number (D', L_a) must be nonnegative for any $a = i, j, k$. Since m_i, m_j, m_k are also nonnegative, formula (44) implies that $(D', L_i) = (D', L_j) = (D', L_k) = 0$ and $m_i = m_j = m_k = 0$. Hence $D = D'$ and $(D, L_i) = (D, L_j) = (D, L_k) = 0$. It follows that D is an effective divisor with $(D, L_a) = 0$ not containing L_a as its irreducible component for every $a = i, j, k$. This means that the compact curve D does not intersect $L = L_i \cup L_j \cup L_k$ and hence must lie in the affine cubic surface $\mathcal{S}(\theta) = \overline{\mathcal{S}}(\theta) - L$. But no compact curve can lie in any affine variety. This contradiction establishes the lemma. $\square$

For each $N \in \mathbb{Z}$ let $\mathit{\Gamma}_N \subset \overline{\mathcal{S}}(\theta) \times \overline{\mathcal{S}}(\theta)$ be the graph of the N-th iterate $c^N : \overline{\mathcal{S}}(\theta) \circlearrowleft$, and $\mathit{\Delta} \subset \overline{\mathcal{S}}(\theta) \times \overline{\mathcal{S}}(\theta)$ be the diagonal. Note that $\mathit{\Gamma}_N = \mathit{\Gamma}_{-N}^{\vee}$, where $\mathit{\Gamma}_{-N}^{\vee}$ is the reflection of $\mathit{\Gamma}_{-N}$ with respect to the diagonal $\mathit{\Delta}$. Moreover let $I_N \subset \overline{\mathcal{S}}(\theta)$ denote the indeterminacy set of c^N. Then the Lefschetz fixed point formula consists of two equations concerning the intersection number $(\mathit{\Gamma}_N, \mathit{\Delta})$ of $\mathit{\Gamma}_N$ and $\mathit{\Delta}$ in $\overline{\mathcal{S}}(\theta) \times \overline{\mathcal{S}}(\theta)$,

$$(\mathit{\Gamma}_N, \mathit{\Delta}) = \sum_{q=0}^{4} (-1)^q \operatorname{Tr}\left[\, (c^N)^* : H^q(\overline{\mathcal{S}}(\theta), \mathbb{Z}) \circlearrowleft \right], \tag{45}$$

$$(\mathit{\Gamma}_N, \mathit{\Delta}) = \#\, \overline{\operatorname{Per}}_N(c) + \sum_{p \in I_N} \mu((p,p), \mathit{\Gamma}_N \cap \mathit{\Delta}), \tag{46}$$

where $\mu((p,p), \mathit{\Gamma}_N \cap \mathit{\Delta})$ denotes the multiplicity of intersection between $\mathit{\Gamma}_N$ and $\mathit{\Delta}$ at (p,p). Lemma 8.1 guarantees that all terms involved in (45) and (46) are well defined and finite.

Lemma 8.2 *Formula (45) becomes* $(\mathit{\Gamma}_N, \mathit{\Delta}) = (2 + \sqrt{5})^N + (2 - \sqrt{5})^N + 4(-1)^N + 2$.

Proof. We put $T_N^q = \operatorname{Tr}\left[\, (c^N)^* : H^q(\overline{\mathcal{S}}(\theta), \mathbb{Z}) \circlearrowleft \right]$. Because $\overline{\mathcal{S}}(\theta)$ is a smooth rational surface,

$$H^q(\overline{\mathcal{S}}(\theta), \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & (q = 0, 4), \\ 0 & (q = 1, 3). \end{cases}$$

Naturally we have $T_N^0 = 1$ and $T_N^1 = T_N^3 = 0$. Since c and so c^N are birational, we have $T_N^4 = 1$. By assertion (4) of Proposition 7.7 the map c is AS, and so Lemma 7.3 implies that $(c^N)^* = (c^*)^N : H^2(\overline{\mathcal{S}}(\theta), \mathbb{Z}) \circlearrowleft$. Recall that c^* has eigenvalues 0, -1 and $2 \pm \sqrt{5}$, where the eigenvalue -1 is quadruple while the remaining ones are simple (see Proposition 7.5). Thus we have $T_N^2 = 0^N + 4(-1)^N + (2+\sqrt{5})^N + (2-\sqrt{5})^N$. Substituting these data into (45) yields the assertion of the lemma. $\square$

Lemma 8.3 *Formula (46) becomes* $(\mathit{\Gamma}_N, \mathit{\Delta}) = \#\, \overline{\operatorname{Per}}_N(c) + 1$ *with* $\#\, \overline{\operatorname{Per}}_N(c) = \#\, \operatorname{Per}_N(c) + 1$.

Proof. By Proposition 7.7, for any $N \in \mathbb{N}$, the point p_k is the unique indeterminacy point of c^N and the point p_i is the unique fixed point of c^N on L. Namely we have $I_N = \{p_k\}$ and $\overline{\operatorname{Per}}_N(c) = \operatorname{Per}_N(c) \cup \{p_i\}$, which implies that formula (46) is rewritten as

$$\begin{aligned} (\mathit{\Gamma}_N, \mathit{\Delta}) &= \#\, \overline{\operatorname{Per}}_N(c) + \mu((p_k,p_k), \mathit{\Gamma}_N \cap \mathit{\Delta}), \\ \#\, \overline{\operatorname{Per}}_N(c) &= \#\, \operatorname{Per}_N(c) + \nu(p_i, c^N), \end{aligned} \tag{47}$$

where $\nu(p_i, c^N)$ is the local index of the map c^N around the fixed point p_i. By assertion (3) of Proposition 7.7, for any $N \in \mathbb{N}$, the point p_i is a superattracting fixed point of c^N and so

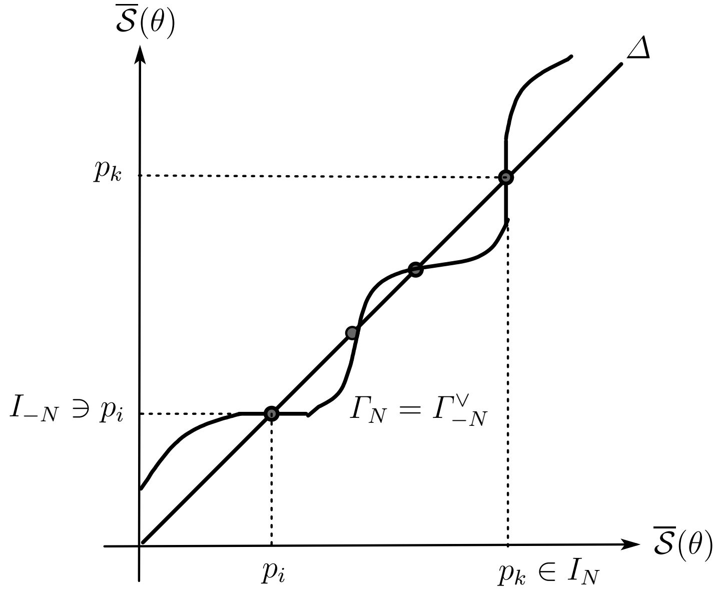

Figure 10: The indeterminacy point p_k of c^N is a superattracting fixed point of c^{-N}

$\det(I - (dc^N)_{p_i}) = \det(I - O) = 1$. This means that $\nu(p_i, c^N) = 1$. Likewise, since p_k is a superattracting fixed point of $c^{-N} = (c^{-1})^N$ where $c^{-1} = \sigma_k \circ \sigma_j \circ \sigma_i$ (see Figure 10), the same reasoning as above with c replaced by c^{-1} yields $\nu(p_k, c^{-N}) = 1$. Therefore we have

$$\mu((p_k, p_k), \Gamma_N \cap \Delta) = \mu((p_k, p_k), \Gamma_{-N}^{\vee} \cap \Delta) = \mu((p_k, p_k), \Gamma_{-N} \cap \Delta) = \nu(p_k, c^{-N}) = 1.$$

These arguments imply that (47) is equivalent to the statement of the lemma. □

Putting Lemmas 8.2 and 8.3 together, we have established the following theorem.

Theorem 8.4 *For any $N \in \mathbb{N}$, the cardinalities of periodic points of period N are give by*

$$\begin{aligned} \#\,\overline{\mathrm{Per}}_N(c) &= (2+\sqrt{5})^N + (2-\sqrt{5})^N + 4(-1)^N + 1, \\ \#\,\mathrm{Per}_N(c) &= (2+\sqrt{5})^N + (2-\sqrt{5})^N + 4(-1)^N. \end{aligned} \tag{48}$$

Then our main theorem (Theorem 2.1) is an immediate consequence of (41) and the second formula of (48). Thus the proof of Theorem 2.1 has just been completed.

In this article we have seen that geometry of cubic surfaces and dynamics on them played an important part in understanding an aspect of the global structure of the sixth Painlevé equation. Their relevance to other aspects will be explored elsewhere.

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